AN ARM TRICK FOR UNARMED PERFORMERS

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One of the most popular facts in mathematics surely is that the fundamental group (invented by Poincaré in the 1890's) of the three-dimensional rotation group $SO(3)$ has just two elements. So two times the generator – represented by a loop in the space of rotations – is zero, and you can entertain your non-mathematical (sometimes perhaps also mathematical) friends or less experienced students, for instance, with the soup-plate trick (a simple description can be found in 8.10.3 of M. Berger's Géométrie. 1. Cedic/Nathan, Paris 1977), if you are "armed" with a plate. But what if you are, so to say, an unarmed person, with no plate, no similar object handy? Is there any chance that even then you can entertain your audience, using the fact mentioned above? The answer is positive, as you will see below.

An arm trick. You are invited to do the following experiment. To describe it unambiguously, I shall speak about your right arm.

STEP 1. Relax your arms.

STEP 2. Stand firmly, looking (for instance) to a wall in front of you, so that your back is parallel to the wall. During the experiment, you will move the right arm, but the rest of your body should remain fixed (if possible, support your back by some vertical object).

Put your right arm out straight horizontal, with the palm in the vertical position (hence your thumb is at the top, and the little finger is at the bottom), with all the fingers tight and straight, so that the end of your longest finger is softly touching the wall.

STEP 3. Keeping your back always parallel to and at a constant distance from the wall and keeping your right arm always straight, with all the fingers tight and straight, slowly rotate your arm, in the sense downwards-backwards-upwards-forwards, alongside your body (roughly speaking, in a plane perpendicular to the wall) so that the end of your longest finger draws (in the air) one complete circle. After the rotation, your palm is again to be in the vertical position. Of course, on doing the rotation as described, there comes a moment, on going backwards with your hand, when you are no longer able to keep your palm in the plane perpendicular to the wall: you feel a torsion in the muscles of your rotating arm and you are pressed to rotate the palm, eventually by 360 degrees, the axis of the palm-rotation passing along your straight arm.

What do you find after the rotation? You feel that your muscles remain in a certain torsion, tension, somehow contracted. Even more: after having done the rotation, you see that your arm became shorter, by *some centimeters*!

STEP 4. After completing the rotation from Step 3, do such a rotation immediately for a second time. What a surprise! After the second rotation, your muscles are not any longer in torsion; they are again normalized. Even more: after having done this second rotation, you see that your arm has again the same length as prior to the experiment!
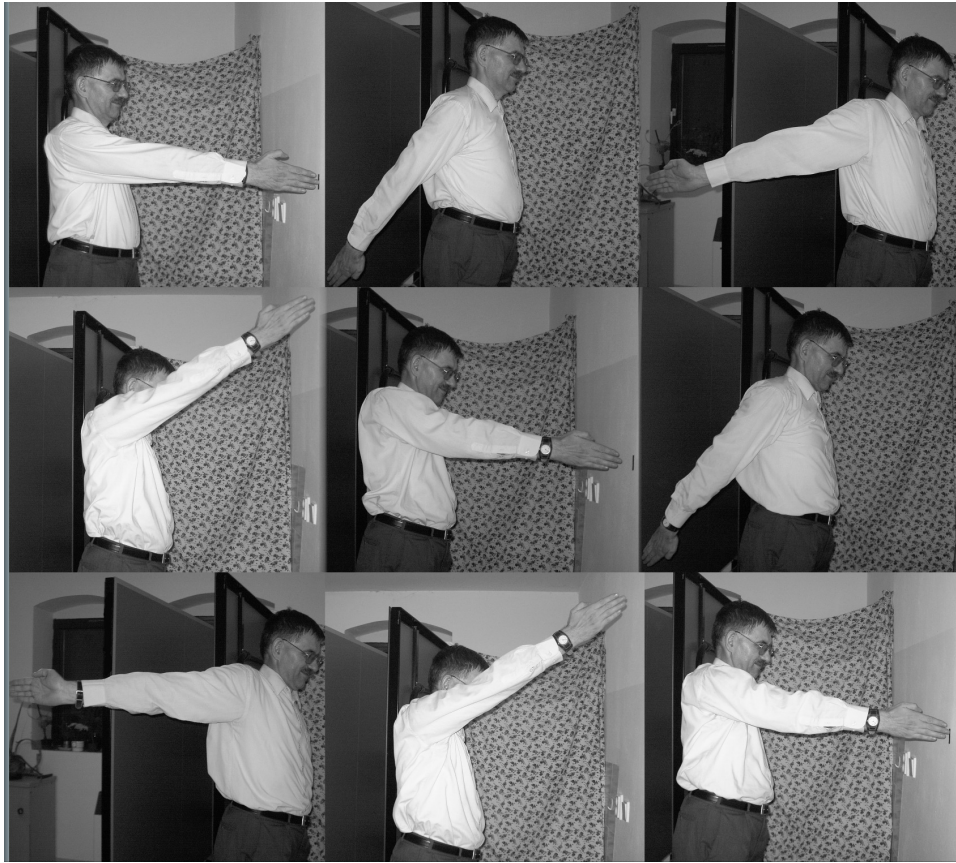
FIGURE 1. Two consecutive rotations (photographed by Rafael Korbaš)

The experiment is illustrated by Figure 1. It consists of nine partial photos, placed in three rows and three columns; the partial photo in the ith row and jth column is referred to as (i, j). For the first rotation, see $(1,1) \rightarrow (1,2) \rightarrow (1,3) \rightarrow (2,1) \rightarrow (2,2)$; for the second rotation, see $(2,2) \rightarrow (2,3) \rightarrow (3,1) \rightarrow (3,2) \rightarrow (3,3)$.

If you succeeded in the experiment, congratulations, you can start looking for a suitable audience to share your joy and teach others. If you tried it many times, strictly following our description, analysing your possible faults, and in spite of all you did not observe the desired phenomena, then you are free to blame the author. But please do not be excessively cruel: if nothing else, you can perhaps recognize that by doing the arm-rotations, you followed your doctor's long-standing advice to take more exercise!

A mathematical explanation of the arm trick. As indicated in the very beginning, our experiment has the same background as the soup-plate trick (or similar tricks, like the Philippine wine-glass dance or the spinor spanner; about the latter two, I was told by Jim Stasheff to whom I am grateful for his encouragement). Indeed, the first rotation of your arm realizes a loop r in $SO(3)$, that is, in the

space of rotations of $\mathbb{R}^3$. Denoting by I the 0-degree rotation, the loop r is not homotopic (i.e., not suitably deformable) to the constant loop at I (that is why you feel a torsion or contraction in the muscles after the first rotation), and therefore represents the generator $[r]$ of the fundamental group $\pi(SO(3); I) = \mathbb{Z}/2\mathbb{Z}$. Since $2[r] = 0$ in $\pi(SO(3); I)$, the two consecutive rotations as described above give the concatenation of the loop r with itself, $r * r$. Of course, since $[r * r] = 2[r] = 0$, the loop $r * r$ must then be homotopic to the constant loop at I (that is why you feel your muscles again relaxed, after the second rotation).

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